11th tutorial

Reductions. Below you will find a "catalogue" of the most famous problems which are "NP-complete" (nevermind what that really means, just know that they all reduce to each other). Try to find the reductions between them. Some of those reductions are hard, some are easy :) A few suggestions of what to try:

(1) Independent Set and Vertex Cover
(2) Subset Sum and 2-Partitions (both directions)
(3) Subset Sum to Knapsack
(4) 3D-matching to 0/1 linear equalities.
(5) 0/1 linear equalities to Subset Sum. Hint: multiplying a matrix by a binary vector x corresponds to a sum of columns.

Decisions problems are enough. Show that it is enough to efficiently solve a decision problem to actually find a solution (in poly time). Specifically:

(1) You have a black box which answers a query: is a given formula satisfiable? How can you find a satisfying assignment using this box? (You can query it multiple times.)
(2) You have a black box solving Independent Set, i.e., it answers whether (G, k) is a yes instance, i.e., whether G has an independent set of size $\geq k$. How to use this box to find such an independent set?

2-SAT. Show that 2-SAT is solvable in polynomial time.

Catalogue of NP-complete problems

- *Logical problems:*
  - SAT: satisfiability of CNF formulas
  - 3-SAT: each clause has at most 3 literals
  - 3,3-SAT: moreover each variables appears at most 3 times
  - SAT for general formulas: not just CNF
  - Circuit SAT: satisfiability of a boolean circuit (has 1 output, can you make the output 1)?
- *Graph problems:*
  - Independent Set: does G have an independent set of size at least k?
  - Clique: does G have a complete subgraph of size at least k?
  - Vertex Cover: is there a subset of vertices U such that each edge has at least one endpoint in U?
  - k-colorability: can G be colored with k colors, such that no two neighbors have the same color? (hard for any $k \geq 3$)
  - Hamiltonian Path: does G contain a path on n vertices (all vertices)
  - Hamiltonian uv-path: does G contain a uv-path containing all vertices?
  - Hamiltonian cycle: does G contain a cycle containing all vertices?
  - Travelling salesman problem: edges have lengths $\ell(e) \geq 0$, does G contain a hamiltonian circuit of length at most k?
  - 3D-matching: (in the lecture)
- *Numerical problems:*
  - Subset Sum: does a given set $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{N}$ of numbers contain a subset $B \subseteq A$ with a given sum k?
  - Knapsack: you are given items with weights and values, and a capacity of a knapsack. Is there a subset of items of total value at least C, whose weight does not exceed the capacity of the knapsack?
  - 2-Partition is it possible to partition a given set of numbers into two subsets with the same sum?
  - 0/1 linear equations: you are given a matrix $\mathbf{A} \in \{0,1\}^{m \times n}$. Is there a vector $\mathbf{x} \in \{0,1\}^n$ such that $\mathbf{Ax}$ is equal to the vector of all ones?