Reductions. Below you will find a "catalogue" of the most famous problems which are NP-complete.

(1) Reduce 0/1 LINEAR EQUALITIES to SAT
(2) Show that HAMILTONIAN CYCLE remains NP-complete even on graphs of degree at most 3. *Hint:* Reduce (general) HAMILTONIAN PATH to CUBIC HAMILTONIAN PATH (the graph must have max degree 3)
(3) QUADRATIC EQUATIONS: reduce SAT to the problem of finding a satisfying assignment to a set of quadratic equations in n **real** variables, i.e. each equation is of the form $\sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} x_i x_j + \sum i = 1^n \gamma_i x_i + \delta = 0$. (Notice that it's not even clear this problem belongs to NP, but your task is only to reduce SAT to it, i.e. show that QUADRATIC EQUATIONS is NP-hard.)

P-complete? Say that we defined a problem to be "P-complete" analogously to our definition of NP-completeness (recall that definition!). Which problems in P are P-complete?

Self-reducibility – Decisions problems are enough. Show that it is enough to efficiently solve a decision problem to actually find a solution (in poly time) for the following problems. (One calls this "self-reduction" because you reduce the problem of finding a solution to the problem of deciding whether on exists.)

(1) 3-COLORABILITY
(2) HAMILTONIAN PATH
(3) SUBSET SUM
(4) $\mathbf{Ax} = \mathbf{1}$

Vertex Cover on Trees. Show that the VERTEX COVER problem is polynomially solvable on trees. What if we're interested in the weighted version of the problem – each vertex has a weight, and we want to find a vertex cover of minimum possible weight?

CATALOGUE OF NP-COMPLETE PROBLEMS

- *Logical problems:*
  - SAT: satisfiability of CNF formulas
  - 3-SAT: each clause has at most 3 literals
  - 3,3-SAT: moreover each variables appears at most 3 times
  - SAT FOR GENERAL FORMULAS: not just CNF
  - CIRCUIT SAT: satisfiability of a boolean circuit (has 1 output, can you make the output 1)?
- *Graph problems:*
  - INDEPENDENT SET: does G have an independent set of size at least k?
  - CLIQUE: does G have a complete subgraph of size at least k?
  - VERTEX COVER: is there a subset of vertices U such that each edge has at least one endpoint in U?
  - k-COLORABILITY: can G be colored with k colors, such that no two neighbors have the same color? (hard for any $k \geq 3$)
  - HAMILTONIAN PATH: does G contain a path on n vertices (all vertices)
  - HAMILTONIAN uv-PATH: does G contain a uv-path containing all vertices?
  - HAMILTONIAN CYCLE: does G contain a cycle containing all vertices?
  - TRAVELLING SALESMAN PROBLEM: edges have lengths$\ell(e) \geq 0$, does G contain a hamiltonian circuit of length at most k?
  - 3D-MATCHING:
- *Numerical problems:*
  - SUBSET SUM: does a given set of numbers contain a subset with a given sum?
  - KNAPSACK: you are given items with weights and values, and a capacity of a knapsack. Is there a subset of items of total value at least C, whose weight does not exceed the capacity of the knapsack?
  - 2-PARTITION is it possible to partition a given set of numbers into two subsets with the same sum?
  - 0/1 LINEAR EQUATIONS: you are given a matrix $\mathbf{A} \in \{0,1\}^{m \times n}$. Is there a vector $\mathbf{x} \in \{0,1\}^n$ such that $\mathbf{Ax}$ is equal to the vector of all ones?