9th tutorial

Definition (*Discrete Fourier Transform (DFT)*): Fix ω to be a primitive n-th root of 1. DFT is the mapping $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$ defined as $\mathbf{y} = \mathcal{F}(\mathbf{x})$, $y_j = \sum_{k=0}^{n-1} x_k \omega^{jk}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Notice that in FFT, we are computing the graph representation of $\mathbf{p} = (p_0, \ldots, p_{n-1})$ with respect to the points $\omega^1, \omega^2, \ldots, \omega^n$ which is exactly $\mathcal{F}(\mathbf{p})$.

1. Fourier Images. Compute the Fourier images of the following vectors:

(1) $(x, x, \ldots, x)$ for $x \in \mathbb{R}$ (try first $x = 1$)
(2) $(1, -1, 1, -1, \ldots, 1, -1)$
(3) $(1, 0, 1, 0, \ldots, 1, 0)$
(4) $(\omega^0, \omega^1, \omega^2, \ldots, \omega^{n-1})$
(5) $(\omega^0, \omega^2, \omega^4, \ldots, \omega^{2n-2})$

2. DFT of a Real Vector. Show that the Fourier image $\mathbf{y}$ of a real vector $\mathbf{x}$ is *antisymmetric*, i.e., $\mathbf{y}_j = \overline{\mathbf{y}}_{n-j}$ for all indices j. What will be the Fourier image of an antisymmetric vector?

Definition (*Reduction*): A problem L reduces to a problem M if there exists a polytime function $f : \{0,1\}^* \to \{0,1\}^*$ such that $L(x) = M(f(x))$ for every $x \in \{0,1\}^*$. Intuitively: $L \to M$ if there's a polytime algorithm reducing Yes-instances of L to Yes-instances of M, and No-instances to No-instances.

3. Properties of $\to$. Prove that the "$\to$" relation in the space of problems has the following properties:

- It is *reflexive*: $A \to A$
- It is *transitive*: $A \to B \wedge B \to C \Rightarrow A \to C$
- It is not *antisymmetric* (recall the definition)
- There exist mutually irreducible problems (A does not reduce to B and B does not reduce to A)

4. Easy Problems. Show that the following problems are polynomial-time solvable:

- 2-Coloring (decide whether a graph G can be colored with 2 colors.)
- Does G have a clique of size 42?
- Given a formula in disjunctive normal form (DNF), does it have a satisfying assignment? (An example of a formula in DNF is $(x_1 \wedge \neg x_2 \wedge x_3) \vee (x_2 \wedge x_4 \wedge x_5) \vee (\neg x_1 \wedge \neg x_2 \wedge x_4)$ – notice that it is composed of *clauses* which are a conjunction of literals, and the formula is obtained by joining the clauses with disjunctions.)