9th tutorial

1. Divisibility. Design a network of depth $\mathcal{O}(\log n)$ which decides whether an n-bit number given in binary is divisible by 5.

2. Rectangles. You are given a set of n axis-parallel rectangles; beware, they are allowed to overlap. Compute the total surface of their union.

Definition (*Discrete Fourier Transform (DFT)*): Fix ω to be a primitive n-th root of 1. DFT is the mapping $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$ defined as $\mathbf{y} = \mathcal{F}(\mathbf{x})$, $y_j = \sum_{k=0}^{n-1} x_k \omega^{jk}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Notice that in FFT, we are computing the graph representation of $\mathbf{p} = (p_0, \ldots, p_{n-1})$ with respect to the points $\omega^1, \omega^2, \ldots, \omega^n$ which is exactly $\mathcal{F}(\mathbf{p})$.

3. Fourier Images. Compute the Fourier images of the following vectors:

(1) $(x, x, \ldots, x)$ for $x \in \mathbb{R}$ (try first $x = 1$)
(2) $(1, -1, 1, -1, \ldots, 1, -1)$
(3) $(1, 0, 1, 0, \ldots, 1, 0)$
(4) $(\omega^0, \omega^1, \omega^2, \ldots, \omega^{n-1})$
(5) $(\omega^0, \omega^2, \omega^4, \ldots, \omega^{2n-2})$

4. Properties. Which properites are expressed by the 0-th and the $n/2$-th coefficient of the Fourier image?

5. Basis Image. What is the Fourier image of the unit vector $\mathbf{e}_j$, i.e., the vector whose j-th coordinate is 1 and all other coordinates are 0?

6. Basis Inversion. For each j find a vector, whose Fourier image is $\mathbf{e}_j$. How to use this to construct the inversion of $\mathcal{F}$, i.e., $\mathcal{F}^{-1}$?

7. DFT of a Real Vector. Show that the Fourier image $\mathbf{y}$ of a real vector $\mathbf{x}$ is *antisymmetric*, i.e., $vey_j = \overline{\mathbf{y}}_{n-j}$ for all indices j. What will be the Fourier image of an antisymmetric vector?