5. tutorial

Shortest Paths in Graphs without Edge Lengths

1. Broken Taxi. A taxi broke down in Manhattan, so it can only drive straight or turn right. (Manhattan is an $n \times n$ grid.) You are given the location of the taxi, the location of a repair shop, and your task is to give instructions to the driver on how to reach the shop and burn as little gas as possible.

2. Ghost. There is a maze and at position S a ghost. Think of the maze as a grid where each cell is either a hallway or a wall. The ghost can walk through walls, but it moves through walls $4\times$ slower than through hallways. The ghost wants to reach a position T. Find the shortest path.

3. Guard. We have a maze on an $N \times N$ grid with walls, a hero, and a treasure. We want to find a way for the hero to reach the treasure as quickly as possible, but not run into a guard (that is, appear in the same cell as the guard at the same time). The guard has a fixed route of length L as a sequence of adjacent cells and it goes back and forth on this route. How to find such a shortest route as quickly as possible? What is the complexity of your algorithm in terms of N and L?

4. Silly Robot. There is a grid maze with hallways and walls. In it, a robot which goes straight and only turns if it hits a wall or the boundary of the maze (i.e., reaches a cell adjacent to a wall, and reaches this cell from a direction orthogonal to the wall). You select the direction in which it turns (left, right, backwards). Find a path from a start cell s to a target cell t with the least number of turns.

5. Lost Robots. There is a maze and 2 robots at different places in the maze. However, these robots are controlled by the same remote control, which has four buttons – going north, south, east, and west. When a robot receives a command which cannot be executed, it ignores it. How to find a sequence of commands which takes both robots out of the maze? (Once a robot is out of the maze, it stops listening to commands.) How to find the shortest such sequence?

6. Few flights. Let's have a map with vertices representing cities and "type 1" directed edges representing roads, and "type 2" edges representing flights. Assume that gas and car rental are cheap (haha), but flying is expensive, so you can afford at most k flights. Is there a city from which we can start and get to any other city with at most k flights?

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7. Many paths. Construct a graph on n vertices with vertices s, t which has $2^{\Omega(n)}$ shortest paths from s to t.

8. Dijkstra fail. Construct a graph with integral (possibly negative) lengths, but without a negative cycle, where Dijkstra's algorithm would run exponentially long. (This is the variant of the algorithm which, if it processes a vertex v and changes the estimate for its neighbor u, it opens the neighbor even if it was previously closed, that is, puts it in the priority queue even if it extracted it previously.)

9. Negative edges $+k$? Can we get rid of negative edges by adding some number $k \in \mathbb{N}$ to all edge lengths, so that we can use Dijkstra's algorithm to find the shortest path even in a graph with negative edges?

10. Likeliest path. Imagine a computer network described by a directed graph, whose vertices correspond to routers and edges are links between them. For each link we have a probability of it being functional. The probability that a certain path is functional is the product of the probabilities of individual links along this path. How to find a path between two routers which is least likely to fail (most likely to work)?

11. Cheapest of shortest. Roads on a map are labeled with two numbers: length and toll. How to find the cheapest among the shortest paths?

12. Max-min tunnel. Let's have a map of a city represented by directed graph. Each edge is labeled by its clearance – what is the highest truck which can pass on this road? That is, given a path, the maximum height of a truck which can pass through the path is determined by the minimum clearance along the path. Given two vertices s, t, how to find a path which allows the tallest load to pass through (i.e., a path with the maximum minimum clearance)?