Elections, Bribery, and Integer Programming

Martin Koutecký

TECHNION
Israel Institute
of Technology

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Outline

1. Elections & Bribery: Geometric Viewpoint

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2. New Algorithms for Bribery

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2. New Algorithms for Bribery
3. Modeling Campaigning: Polytope Games ($\Rightarrow$ open problems)

Elections & Bribery: Geometric Viewpoint

Computational Social Choice

- Ancient questions
  - Who should govern?
  - How to select them?
  - What is good for society?
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- Recent topic
  - Brexit
  - Trump
  - Facebook

Voting

Candidates: ▲, ■, and ★.
People: preference (e.g. ■ $\succ$ ▲ $\succ$ ★), active/latent, bribery costs, etc.
(simplify: just preference)
Society: how many people of which type $\Rightarrow$ **Society graph:**

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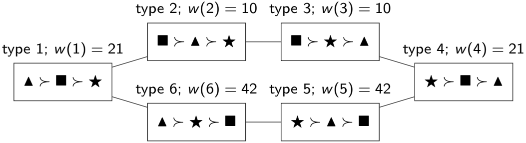

Society $\mathbf{w} = (21, 10, 10, 21, 42, 42)$
edges $\equiv$ swap distance 1.

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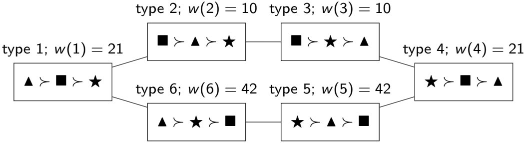

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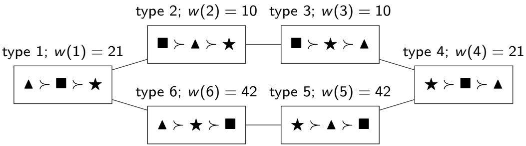

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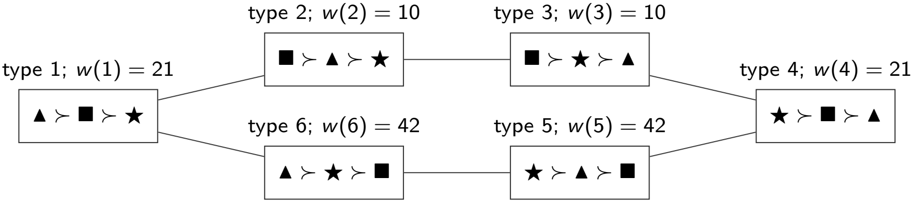

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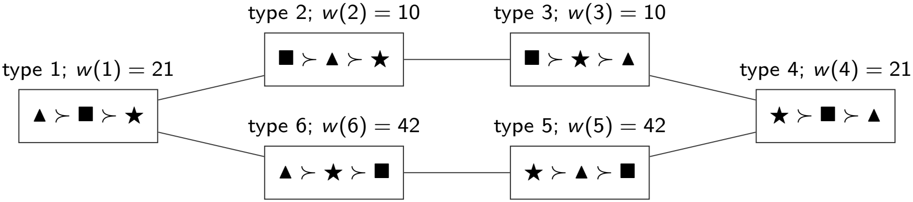

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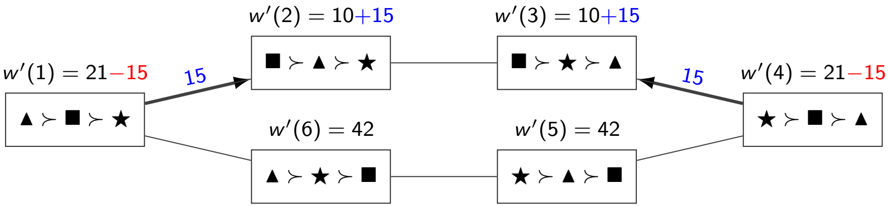

society $\mathbf{w} = (21, 10, 10, 21, 42, 42)$
move $\mathbf{m} = (0, \ldots, 0, +15, +15, 0, \ldots, 0)$ (arc space)
change $\boldsymbol{\Delta} = \Delta(\mathbf{m}) = (-15, +15, +15, -15, 0, 0)$

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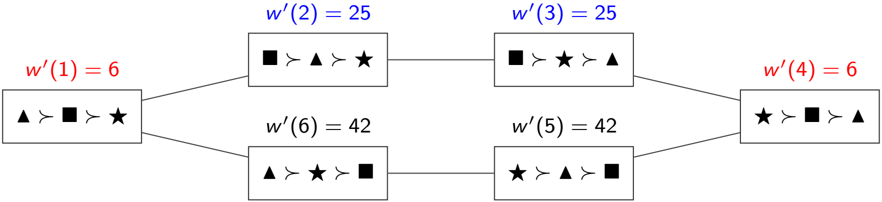

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■ **wins:** $48 = w(1) + w(6) = w(4) + w(5) < w(2) + w(3) = 50$

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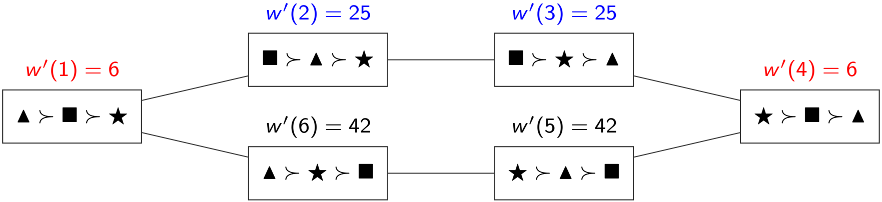

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Bribery: cheapest way to move voters s.t. $\blacksquare$ wins Plurality?
(Assume unit cost per swap.)
Robust model: captures many prior manipulation models – full bribery, only shift $\blacksquare$, pay-per-swap, add/delete voters, etc.

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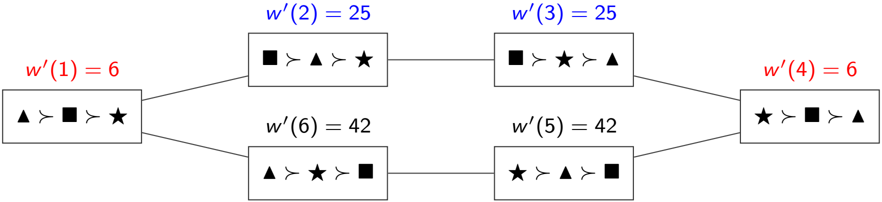

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BTW: Society graph + move + change model is “obvious” but new and very useful! [IJCAI; Faliszewski, Gonen, K., Talmon] and [AAMAS; Knop, K., Mnich]

New Algorithms for Bribery

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Challenge #1: Replace Lenstra, make single-exp!
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[2014; Bredereck, Chen, Faliszewski, Guo, Niedermeier, Woeginger]

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Solved!

Theorem (STACS, ESA, AAMAS; Knop, K., Mnich)

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$$\min \mathbf{cm} \qquad \text{minimum cost move}$$

$$\mathbf{w}' = \mathbf{w} + \Delta(\mathbf{m}) \geq \mathbf{0} \qquad \text{move } \mathbf{m} \text{ produces a valid society } \mathbf{w}'$$

$$\sum_{i:\text{type } i \text{ votes for } c} w'_i = S_c \qquad \forall c \qquad \text{aggregate points}$$

$$S_{c^\star} \geq S_c \qquad \forall c \neq c^\star \qquad c^\star \text{ wins}$$

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$\mathcal{O}(|C|!)$ variables $\Rightarrow$ apply Lenstra: $|C|!^{|C|!}\langle \mathbf{w} \rangle = 2^{2^{\#\text{candidates}^{\mathcal{O}(1)}}} \log(\#\text{people})$

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Proof of (1).

Idea: ILP has *n*-fold structure!
Blocks $\sim$ types of people,
A block $\sim$ #ppl moving to other type,
$(D \cdots D) \sim$ voting rule.

$$\begin{pmatrix} D & D & \cdots & D \\ \mathbf{1} & 0 & \cdots & 0 \\ 0 & \mathbf{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} \end{pmatrix}$$

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Adapt/extend algo [Hemmecke, Onn, Romanchuk '13]
"simple rules": *few* constraints, *small* $\|D\|_\infty$.

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Want: formula $\Phi_{\text{Dodgson}} \equiv$ "★ is Dodgson winner" $\equiv$ least #swaps to Condorcet

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Encode Φ_{Dodgson} in terms of society / move / change vectors
$\Rightarrow$ decide $\exists \mathbf{x} \forall \mathbf{y} \exists \mathbf{z} : \Psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ sentence $\Rightarrow$ [much modeling work]
$\Rightarrow$ decide $\forall \mathbf{x} \exists \mathbf{y} : A(\mathbf{x}, \mathbf{y}) \leq \mathbf{b}$ sentence

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Thm [Eisenbrand, Shmonin '08]: Can decide $\forall \mathbf{b} \in Q \cap \mathbb{Z}^m \exists \mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} \leq \mathbf{b}$
in time $f(n, m) \cdot \text{poly}(\|A, \mathbf{b}\|_\infty)$ □

Modeling Campaigning: Polytope Games

Campaigning Game

So far: bribe, then vote. *What about more rounds?*

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k-**round Campaigning Game:**

Given:

- Society $\mathbf{w}^0$,
- cost vectors $\mathbf{c}^1, \ldots, \mathbf{c}^{2k}$,
- round budgets $b^1, \ldots, b^{2k} \in \mathbb{N}$,
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Decide: P has winning strategy?

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$k \geq 2 \equiv \exists \mathbf{x}^1 \cdots \forall \mathbf{y}^k : A(\mathbf{x}^1, \ldots, \mathbf{y}^k) \leq \mathbf{b}$

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- Q reacts by picking $\mathbf{m}^{2i}$ with cost $\mathbf{c}^{2i}\mathbf{m}^{2i} \leq b^{2i}$ valid for society $\mathbf{w}^{2i-1}$
- $\Rightarrow \mathbf{w}^{2i} := \mathbf{w}^{2i-1} + \Delta(\mathbf{m}^{2i-1})$

P **wins** if $\bigstar$ wins in society $\mathbf{w}^{2k}$ under rule $\mathcal{R}$.

Decide: P has winning strategy?

Campaigning Game

So far: bribe, then vote.

What about more rounds?

k-**round Campaigning Game:**

Given:

- Society $\mathbf{w}^0$,
- cost vectors $\mathbf{c}^1, \ldots, \mathbf{c}^{2k}$,
- round budgets $b^1, \ldots, b^{2k} \in \mathbb{N}$,
- voting rule $\mathcal{R}$,
- players P and Q.

$k = 1 \equiv \exists \mathbf{x} \forall \mathbf{y} : A(\mathbf{x}, \mathbf{y}) \leq \mathbf{b} \Rightarrow$
solvable in $f(\#\text{types})\,\text{poly}(\#\text{people})$
$k \geq 2 \equiv \exists \mathbf{x}^1 \cdots \forall \mathbf{y}^k : A(\mathbf{x}^1, \ldots, \mathbf{y}^k) \leq \mathbf{b}$
BUT

Theorem (Nguyen, Pak '17)

$\forall\exists\forall$-ILP is NP-*c in dimension* ≥ 6*!*

Play: in each round $i \in \{1, \ldots, k\}$,

- P picks a move $\mathbf{m}^{2i-1}$ w/ cost $\mathbf{c}^{2i-1}\mathbf{m}^{2i-1} \leq b^{2i-1}$ valid for society $\mathbf{w}^{2i-2}$
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P **wins** if $\bigstar$ wins in society $\mathbf{w}^{2k}$ under rule $\mathcal{R}$.

Decide: P has winning strategy?

Polytope Game(s)

***k*-round Polytope Game:**

Given:

- Point $\mathbf{x}^0 \in \mathbb{R}^n$,
- polytopes $P_1, Q_1, \ldots, P_k, Q_k \subseteq \mathbb{R}^n$,
- target polytope $W \subseteq \mathbb{R}^n$,
- players P and Q.

Polytope Game(s)

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- polytopes $P_1, Q_1, \dots, P_k, Q_k \subseteq \mathbb{R}^n$,
- target polytope $W \subseteq \mathbb{R}^n$,
- players P and Q.

Play: in each round $i \in \{1, \dots, k\}$,

- P picks a move $\mathbf{p}^i \in P_i$
- $\Rightarrow \mathbf{x}^{2i-1} := \mathbf{x}^{2i-2} + \mathbf{p}^i$
- Q reacts by picking $\mathbf{q}^i \in Q_i$
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Decide: P has winning strategy?

Positions in W "resistent" to moves in Q_k:

$$W' = \{\mathbf{w} \mid \forall \mathbf{q} \in Q_k : \mathbf{w} + \mathbf{q} \in W\} = W \sim Q_k \Leftarrow \text{Minkowski difference!}$$

Polytope Game(s)

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Decide: P has winning strategy?

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$W' = \{\mathbf{w} \mid \forall \mathbf{q} \in Q_k : \mathbf{w} + \mathbf{q} \in W\} = W \sim Q_k \Leftarrow$ Minkowski difference!

Positions in W' "reachable" by moves in P_k:

$W'' = \{\mathbf{w} + \mathbf{p} \mid \mathbf{p} \in P_k, \mathbf{w} \in W'\} = W + P_k \Leftarrow$ Minkowski sum!

Polytope Game(s)

k-round Polytope Game:

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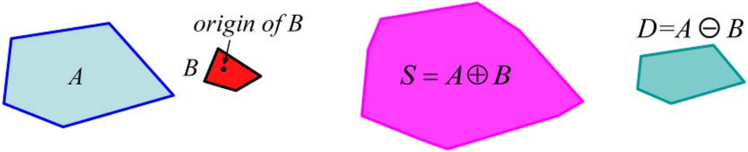
Decide: P has winning strategy?

Positions in W “resistent” to moves in Q_k:

$W' = \{\mathbf{w} \mid \forall \mathbf{q} \in Q_k : \mathbf{w} + \mathbf{q} \in W\} = W \sim Q_k \Leftarrow$ Minkowski difference!

Positions in W' “reachable” by moves in P_k:

$W'' = \{\mathbf{w} + \mathbf{p} \mid \mathbf{p} \in P_k, \mathbf{w} \in W'\} = W + P_k \Leftarrow$ Minkowski sum!



Polytope Game(s)

Positions in W "resistent" to moves in Q_k:

$W' = \{\mathbf{w} \mid \forall \mathbf{q} \in Q_k : \mathbf{w} + \mathbf{q} \in W\} = W \sim Q_k \Leftarrow$ Minkowski difference!

Positions in W' "reachable" by moves in P_k:

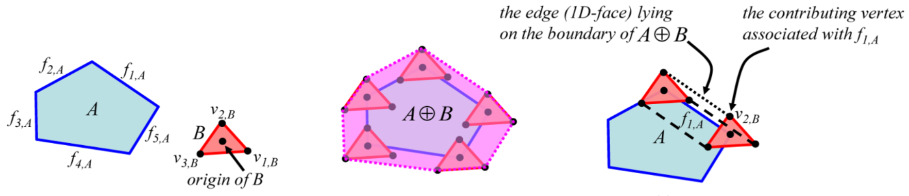
$W'' = \{\mathbf{w} + \mathbf{p} \mid \mathbf{p} \in P_k, \mathbf{w} \in W'\} = W + P_k \Leftarrow$ Minkowski sum!

"Theorem"

Solve in time $f(n, d, k) \cdot \langle \sum_i \|P_i, Q_i\|_\infty \rangle$ *w/* $d = \max_i$ *#ineqs describing* P_i, Q_i.

Proof.

Repeatedly apply Minkowski difference and sum (we always stay convex).



(a) (b) (c)

$\square$

Polytope Game(s)

k-**round Integer Polytope Game:**

Given:

- Point $\mathbf{x}^0 \in \mathbb{Z}^n$,
- polytopes $P_1, Q_1, \ldots, P_k, Q_k \subseteq \mathbb{R}^n$,
- target polytope $W \subseteq \mathbb{R}^n$,
- players P and Q.

Play: in each round $i \in \{1, \ldots, k\}$,

- P picks a move $\mathbf{p}^i \in P_i \cap \mathbb{Z}^n$
- $\Rightarrow \mathbf{x}^{2i-1} := \mathbf{x}^{2i-2} + \mathbf{p}^i$
- Q reacts by picking $\mathbf{q}^i \in Q_i \cap \mathbb{Z}^n$
- $\Rightarrow \mathbf{x}^{2i} := \mathbf{x}^{2i-1} + \mathbf{q}^i$

P **wins** if $\mathbf{x}^{2k} \in W$

Decide: P has winning strategy?

Polytope Game(s)

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Given:

- Point $\mathbf{x}^0 \in \mathbb{Z}^n$,
- polytopes $P_1, Q_1, \ldots, P_k, Q_k \subseteq \mathbb{R}^n$,
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Play: in each round $i \in \{1, \ldots, k\}$,

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Decide: P has winning strategy?

Gets more complicated: even if W', P_k are integer points of a convex set, $W' + P_k$ is not!

Still: given $ILP(W)$ and $ILP(Q_k)$,

can define $ILP(W') = ILP(W) \sim ILP(Q_k)$, and,

can define $ILP(W'') = ILP(W') + ILP(P_k)$.

Polytope Game(s)

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can define $ILP(W') = ILP(W) \sim ILP(Q_k)$, and,

can define $ILP(W'') = ILP(W') + ILP(P_k)$.

"Theorem"

Solve in time $f(k, n, d \max_i \mathit{coeff}(P_i, Q_i)) \cdot \langle \sum_i \mathit{rhs}(P_i, Q_i) \rangle$.

Proof.

Repeatedly apply Minkowski difference and sum (but now the result is possibly non-convex BUT is a projection of a convex set) + Integer hull bounds + Lenstra. □

Polytope Game(s)

k-round Nonnegative Integer Polytope Game:

Given:

- Point $\mathbf{x}^0 \in \mathbb{N}^n$,
- polytopes $P_1, Q_1, \ldots, P_k, Q_k \subseteq \mathbb{R}^n$,
- target polytope $W \subseteq \mathbb{R}^n$,
- players P and Q.

Play: in each round $i \in \{1, \ldots, k\}$,

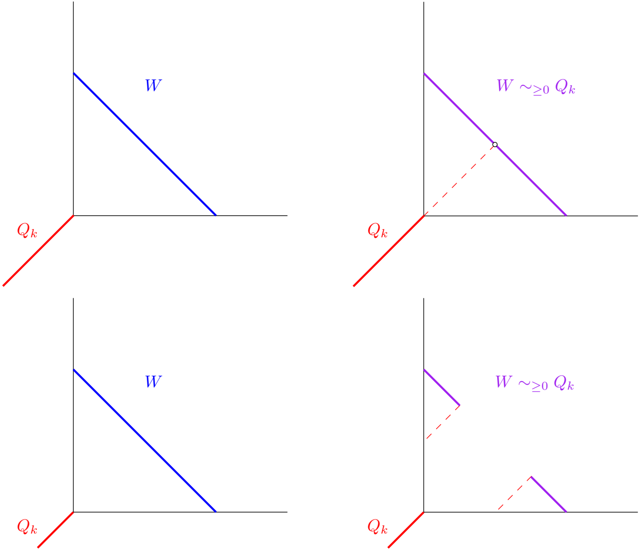
- P picks a move $\mathbf{p}^i \in P_i \cap \mathbb{Z}^n$
- $\Rightarrow \mathbf{x}^{2i-1} := \mathbf{x}^{2i-2} + \mathbf{p}^i \geq \mathbf{0}$
- Q reacts by picking $\mathbf{q}^i \in Q_i \cap \mathbb{Z}^n$
- $\Rightarrow \mathbf{x}^{2i} := \mathbf{x}^{2i-1} + \mathbf{q}^i \geq \mathbf{0}$

P **wins** if $\mathbf{x}^{2k} \in W$

Decide: P has winning strategy?

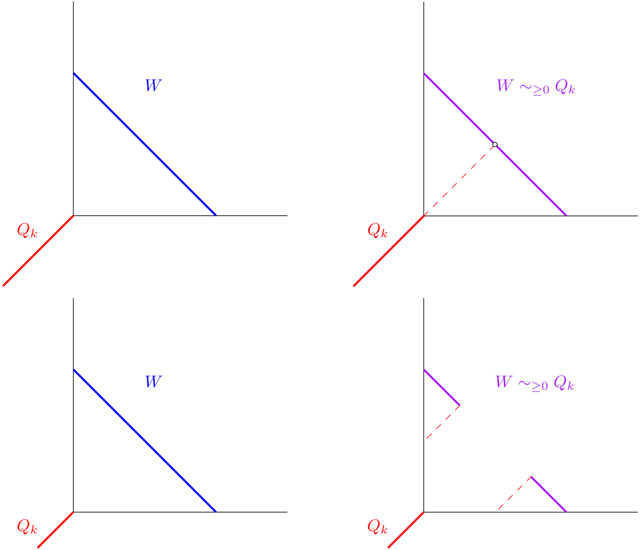
Polytope Game(s)

$W \sim_{\geq 0} Q_k$ not convex anymore, even without integrality!



Polytope Game(s)

$W \sim_{\geq 0} Q_k$ not convex anymore, even without integrality!



What could be done:

- nonnegative game?
- integer nonneg game?

Polytope Game(s)

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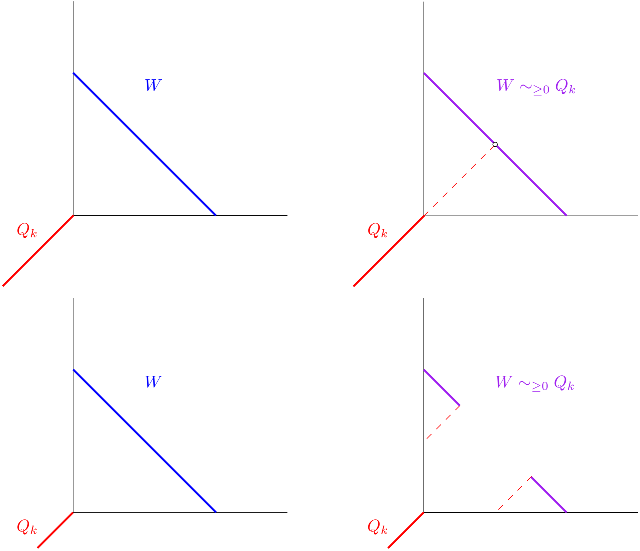

What could be done:

nonnegative game?

integer nonneg game?

Thank you!