An Algorithmic Theory of Integer Programming

Friedrich Eisenbrand, Christoph Hunkenschröder, Kim-Manuel Klein,
Martin Koutecký, Asaf Levin, Shmuel Onn

MIP 2019, MIT

TECHNION
Israel Institute
of Technology

CHARLES UNIVERSITY
Faculty of mathematics
and physics

Outline

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My goal is to sell you this paper:

An Algorithmic Theory of Integer Programming [arxiv:1904.01361]

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Talk Outline:

1. Integer Programming: Structural Parameterizations
2. The Theory: Iterative Augmentation
3. The Extras: Proximity, Scaling, Reducibility, Near-linear / Strongly Polynomial Algorithms, Lower Bounds, etc.
4. Outlook

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Integer Programming: Structural Parameterizations

Variable Dimension: Unifying Theory

👁 **Real world is high-dimensional!**

Brief history of variable dimension IP:

- 1960's: Total Unimodularity (paths, matchings, flows) [Hoffman, Kruskal]
- 1980's: ILPs with few rows (generalized knapsack) [Papadimitriou; Eisenbrand, Weismantel; Jansen, Rohwedder]
- 2010–: Iterative methods for block structured programs [Aschenbrenner, Chen, De Loera, Hemmecke, Köppe, Lee, Marx, Onn, Romanchuk, Schulz, Weismantel]
- 2015–: Tree-structured ILPs [Ganian, Jansen, Kratsch, Ordyniak, Ramanujan]

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No strongly polynomial algorithms for these classes (and few overall: TU, bimodular, binet).

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- 2010–: Iterative methods for block structured programs [augmentation, Ramsey, Algebra, DP]
- 2015–: Tree-structured ILPs [Lenstra, treewidth]

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Our result: improves, unifies, simplifies, makes strongly-poly *all* of these results!

Structural Parameterizations: The Graphs of A

$$\min f(\mathbf{x}) \; : \; A\mathbf{x} = \mathbf{b}, \; \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \; \mathbf{x} \in \mathbb{Z}^n \qquad \text{(IP)}$$

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$$
\begin{aligned}
x_1 + 2x_2 + x_4 &= 11 && (C_1)\\
3x_3 - 2x_4 &= 6 && (C_2)\\
x_2 + 2x_3 &= 2 && (C_3)\\
-x_1 + 3x_4 &= 2 && (C_4)\\
0 \leq x_1, x_2, x_3, x_4 &\leq 5 && \text{(box)}
\end{aligned}
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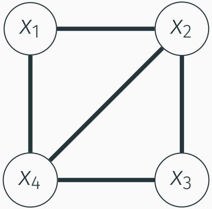
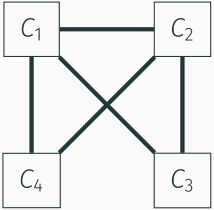
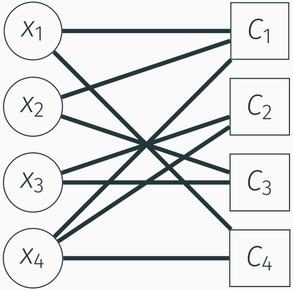


Primal: $G_P(A)$



Dual: $G_D(A)$



Incidence: $G_I(A)$

Structural Parameterizations: Small Graphs $\approx$ Classics

Lenstra '83 $\Rightarrow$ ILP solvable in time $f_P(|G_P|) \cdot \langle A, \mathbf{b}, \mathbf{w} \rangle$

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Parameterized complexity perspective:

- Runtime $f(\alpha) \cdot \text{poly}(\beta)$ with
  parameter $\alpha = |G_P| = n$ and
  input $\beta = \langle A, \mathbf{w}, \mathbf{b} \rangle$

$f(\alpha)\,\text{poly}(\beta)$	clearly better than	$\beta^{f(\alpha)}$
FPT (fixed-parameter tractable)		XP

IP has many natural parameters: dimension n, #rows m, largest coefficient $\|A\|_\infty$, etc.

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because $|G_P| < |G_I|$ also in $f_P(|G_I|) \cdot \langle A, \mathbf{b}, \mathbf{w} \rangle$

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Papadimitriou '81 $\Rightarrow$ ILP solvable in time $f_D(|G_D|, \|A\|_\infty) \cdot n \cdot \langle \mathbf{b}, \mathbf{w} \rangle$

can't parameterize only by $|G_D|$ or $\|A\|_\infty$.

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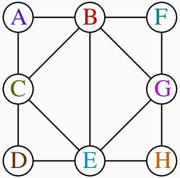
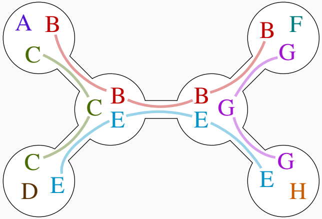
Graph size = too strict a parameter. What else?

Structural Parameterizations: Treewidth





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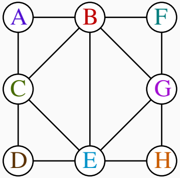

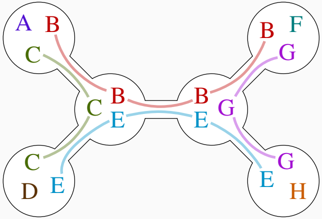

\+ a popular and successful parameter

– not for IP

NP-hard even for $\text{tw}_P, \text{tw}_D, \text{tw}_I \leq 3$, $\|A\|_\infty = 2$

($\text{tw}_P(A) = \text{tw}(G_P(A))$, $\text{tw}_D(A) = \text{tw}(G_D(A))$, $\text{tw}_I(A) = \text{tw}(G_I(A))$.)

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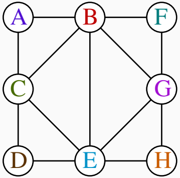

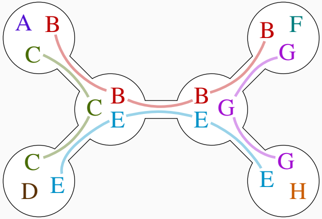

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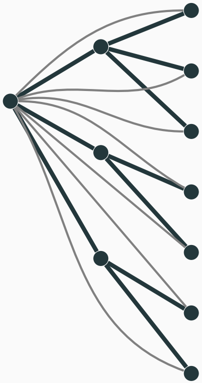
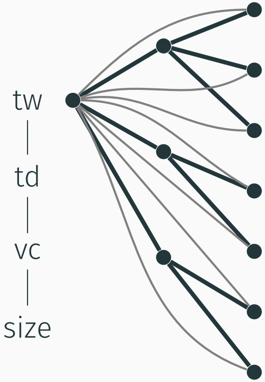
A more restrictive parameter?

Structural Parameterizations: Treedepth

tw
|
td
|
vc
|
size



$$\mathrm{td}(G) = \begin{cases} 1 & \text{if } |V(G)| = 1, \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v) & \text{if connected,} \\ \max_{G_i \text{ component}} \mathrm{td}(G_i) & \text{if disconnected} \end{cases}$$



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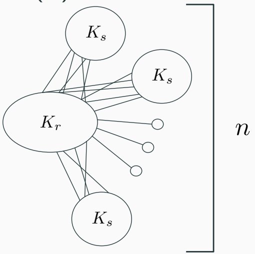
Example: n-fold IP matrix

$$A = \underbrace{\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n}$$

$$\begin{array}{c|c} \overbrace{A_1}^{t} & r \\ A_2 & s \end{array}$$

$G_D(A)$



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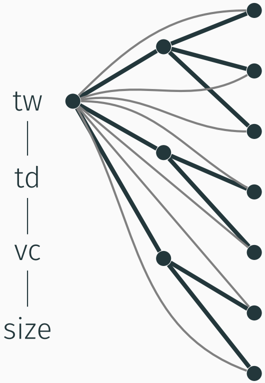

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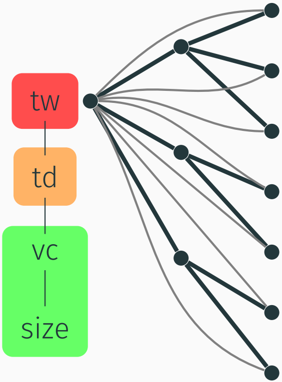
- **Main result: IP solvable in time**
  $g(\min\{\mathrm{td}_P(A), \mathrm{td}_D(A)\}, \|A\|_\infty) \cdot \mathrm{poly}(n)$
  $\Rightarrow$ FPT par. by $\min\{\mathrm{td}_P(A), \mathrm{td}_D(A)\}$ and $\|A\|_\infty$



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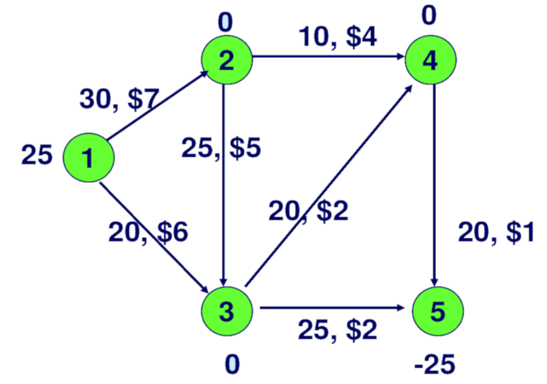
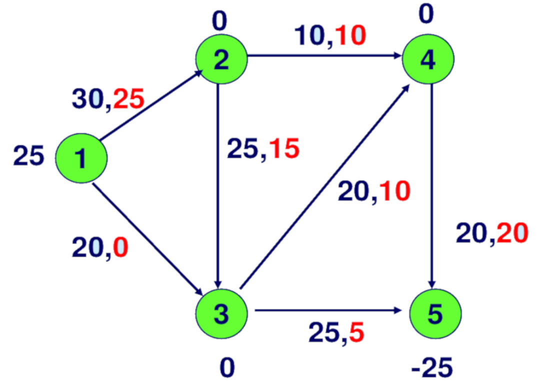
- **Main result: IP solvable in time** $g(\min\{\mathrm{td}_P(A), \mathrm{td}_D(A)\}, \|A\|_\infty) \cdot \mathrm{poly}(n)$
  $\Rightarrow$ FPT par. by $\min\{\mathrm{td}_P(A), \mathrm{td}_D(A)\}$ and $\|A\|_\infty$
- ILP NP-h for $\mathrm{td}_I = 5$ and $\|A\|_\infty = 1$ [Eiben et al. '19]

The Theory: Iterative Augmentation



Min-cost flow



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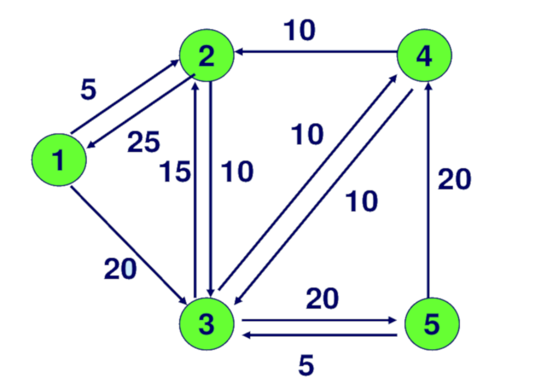

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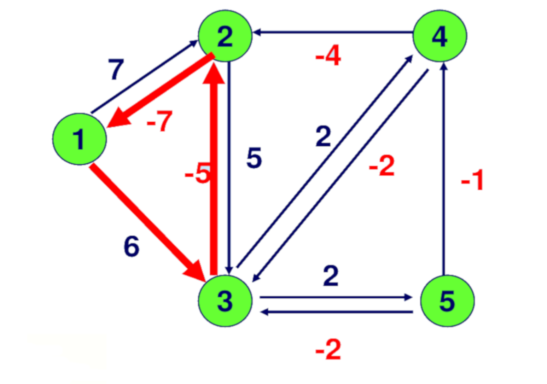

Min-cost flow

a step $\equiv$ a cycle C

(because circulations decompose into cycles)

C *feasible* if enough capacity (fits res. net)

C *augmenting* if negative

flow cost *minimal* if $\nexists$ negative cycle

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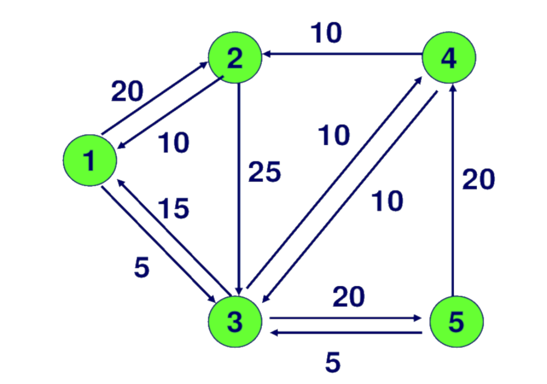

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Integer Programming

$\mathbf{g} \in \mathrm{Ker}_{\mathbb{Z}}(A) = \{\mathbf{g} \in \mathbb{Z}^n \mid A\mathbf{g} = \mathbf{0}\}$
$(A\mathbf{x} = \mathbf{b} \implies A(\mathbf{x} + \mathbf{g}) = \mathbf{b})$

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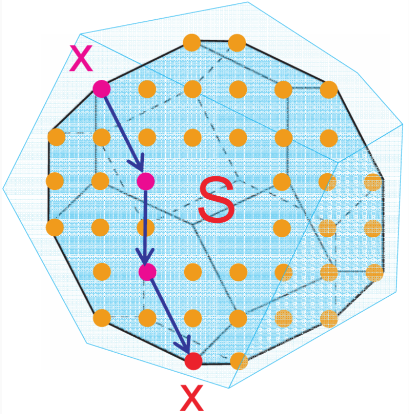

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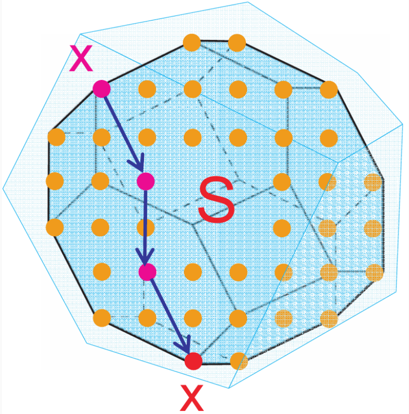

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Goal: Find $\mathcal{T} \subseteq \mathrm{Ker}_{\mathbb{Z}}(A)$, s.t.

1. $\mathbf{x}$ not opt then $\exists$ augmenting $\mathbf{g} \in \mathcal{T}$
2. good convergence for repeatedly adding "good" $\mathbf{g} \in \mathcal{T}$,
3. algorithmically tame (big is OK)

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Answer: ($\mathbf{x} \sqsubseteq \mathbf{y} \Leftrightarrow \mathbf{x}$ and $\mathbf{y}$ in same orthant $\wedge\ |x_i| \leq |y_i|$; $\mathbf{g} \in \mathcal{G}(A) \approx$ "closest to origin")

Definition (Graver basis)

$\mathcal{G}(A) = \{\mathbf{x} \in \mathrm{Ker}_{\mathbb{Z}}(A) \mid \mathbf{x} \text{ is } \sqsubseteq\text{-minimal}\}$

Iterative Augmentation (Sec 2.1)

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Prop 6: Every $\mathbf{h} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ *conformally* decomposes as $\mathbf{h} = \sum_{i=1}^{2n} \lambda_i \mathbf{g}_i$

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Df: $\mathbf{h} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ is *Graver-best step* for $\mathbf{x}$ if as good as $\lambda\mathbf{g}$ for any $\lambda \in \mathbb{N}$, $\mathbf{g} \in \mathcal{G}(A)$

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The Extras

The Extras: Proximity Bounds

Theorem (Basic proximity)

Let $\mathbf{x}^*, \mathbf{z}^*$ *be a fractional and integer optimum, respectively. There exist* $\hat{\mathbf{x}}, \hat{\mathbf{z}}$ *frac/int optima s.t., for any* $p \geq 1$,

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Let $\mathbf{x}^s$ *be opt of* s*-scaled down instance. There exists* $\mathbf{x}^*$ *opt s.t.*

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“Modern version” of [Hochbaum, Shantikumar ’90]

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Goal: Strongly-polynomial algorithm (#arithmetic ops not dep on size of numbers)

1. Solve LP relaxation in $\mathrm{poly}(n \cdot \langle A \rangle)$ time [Tardos '86]
2. Proximity: int opt not far from frac opt $\Rightarrow$ shrink bounds $\mathbf{l}', \mathbf{u}'$, shrink rhs $\mathbf{b}'$.
3. Reduce objective: $\mathbf{l}', \mathbf{u}'$ give small box $\Rightarrow$ equiv. $\mathbf{w}'$ w/ small $\|\mathbf{w}'\|_\infty$
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- Replace halflings with a more expensive, more powerful steps
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Theorem ((more or less))

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Thank you!