12. tutorial

1. Nadiria. In a previous life, you worked as a cashier in the lost Antarctican colony of Nadiria, spending the better part of your day giving change to your customers. Because paper is a very rare and valuable resource in Antarctica, cashiers were required by law to use the fewest bills possible whenever they gave change. Thanks to the numerological predilections of one of its founders, the currency of Nadiria, called Dream-Dollars, was available in the following denominations: \$1, \$4, \$7, \$13, \$28, \$52, \$91, and \$365. ´

(a) Describe and analyze a recursive algorithm that computes, given an integer k, the minimum number of bills needed to make k Dream-Dollars. (Don't worry about making your algorithm fast; just make sure it's correct.)

(b) Describe a dynamic programming algorithm that computes, given an integer k , the minimum number of bills needed to make k Dream-Dollars. (This one needs to be fast.)

2. Hill. Say we have a sequence of integers $a_1, \ldots, a_n$. A *hill* is a subsequence which first increases and then decreases. Design an algorithm, which in a given sequence finds a longest hill (as a subsequence).

3. Library I. Let us have a sequence of n books. Each book has a width w_i and a height h_i. We want to arrange the books in a library with several shelves such that we preserve the alphabetical order of the books. So, the alphabetically first few books go on the first shelf, the next on the second shelf, and so on. We are given a library width W, and we want to arrange the books in shelves in such a way that each book has its place and the total height of the library is as small as possible. (Don't count the height of the shelves.)

4. Marble. You have mined a large slab of marble from a quarry. For simplicity, suppose the marble slab is a rectangle measuring n cm in height and m cm in width. You want to cut the slab into smaller rectangles of various sizes— some for kitchen counter tops, some for large sculpture projects, others for memorial headstones. You have a marble saw that can make either horizontal or vertical cuts across any rectangular slab. At any time, you can query the spot price $P[x, y]$ of an x-cm by y-cm marble rectangle, for any positive integers x and y. These prices depend on customer demand, and people who buy marble counter tops are weird, so don't make any assumptions about them; in particular, larger rectangles may have significantly smaller spot prices. Given the array of spot prices and the integers m and n as input, describe an algorithm to compute how to subdivide an $n \times m$ marble slab to maximize your profit.

5. Transmission. You have decyphered a secret transmission, but the spaces are missing. All is not lost – we have a dictionary containing all the words which may occur in the transmission. The task at hand is thus to split the message into the fewest possible number of dictionary words.
(For example, `ARTISTOIL` can be split as `ART·IS·TOIL`, but splitting it as `ARTIST·OIL` contains fewer words and thus is a better solution.)

6. Library 2. The problem is similar as before. However, now we are given a maximum allowed library height H, and the task is to find the minimum possible width W. If you struggle, start by assuming all books have width 1.

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7. Average Complexity. Prove that $\Omega(\log n)$ and $\Omega(n \log n)$ comparisons are needed for searching and sorting, respectively, not only in the worst case but also on average, where the average is taken over all possible inputs. In the case of search, the average is taken over all possible x being searched; for sorting, the average is taken over all permutations.

8. Matrix search. An $n \times n$ matrix A of integers is given, and it satisfies the property that each row and column forms an increasing sequence. How to quickly find indices i, j such that $A_{i,j} = i, j$? If there are multiple solutions, it suffices to report one of them. We don't count the time needed to load the matrix into memory.
Hint: Look at the lower left corner of the matrix. What is implied by $A_{i,j} < i + j$ or $A_{i,j} > i + j$?